UNIT 2: The Binomial Expansion

2.1 The expansion $(1+x)^n = \sum_{r=0}^{n} C_r^n x^r$ when n is a positive integer.

Students can use the formula to write down the expansion in series without the necessity of doing the actual multiplication. The formal proof of the expansion is not required. Determination of the greatest term and relations between coefficients are excluded. However, the properties of the binomial expansion should include

(i) the expansion contains $n + 1$ terms;

(ii) the binomial coefficients C_r^n; are all integers.

The Pascal triangle should be studied in relation to the coefficients C_r^n; in the expansion.

Students are expected to know

$$\sum_{r=1}^{n}(ax_r \pm by_r) = a\sum_{r=1}^{n} x_r \pm b\sum_{r=1}^{n} y_r$$

$$\sum_{r=1}^{n}\left(x_r \pm y_r\right)^2 = \sum_{r=1}^{n} x_r{}^2 \pm 2\sum_{r=1}^{n} x_r y_r + \sum_{r=1}^{n} y_r{}^2$$

<u>Example 1</u>
Use the binomial expansion to approximate the value of, say, $(1.01)^{10}$.

<u>Example 2</u>
Expansion of multinomials may be obtained by repeated application of the binomial theorem.

<u>Example 3</u>
The application of the binomial expansion in problems involving probability (in Unit 13).

<u>Example 4</u>
The value of the base of the natural logarithm e (Unit 3) is calculated from binomial expansion.

<u>Example 5</u>
The formula of the normal distribution (Unit 14) is calculated from binomial expansion.

2.2 The expansion of $(1+x)^n$ when n is not a positive integer and $|x| < 1$.

Students should learn what happens to the coefficient $\dfrac{n(n-1)\cdots(n-r+1)}{r!}$ of the general term i.e. the (r + l)th term of the expansion.

When n is a positive integer, it vanishes. when $r \geq n + 1$.

If n is not a positive integer, it will never vanish and the expansion is given by
$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots$ provided that $-1 < x < 1$.

This can be illustrated by the following example.

Consider the expansion $(1-x)^{-1} = 1 + x + x^2 + \cdots + x^r + \cdots$.
Putting $x = -1$, we have
R.H.S. = $\begin{cases} 0\ , & \text{even number of terms} \\ 1\ , & \text{odd number of terms} \end{cases}$

But L.H.S. = $[1-(-1)]^{-1} = \frac{1}{2}$

The meaning of the absolute value should also be taught and informal discussion on the intuitive meaning of convergence and divergence may be introduced.

Expansions for particular values of n should be studied, such as

(i) $(1+x)^{\frac{1}{3}} = 1 + \frac{x}{3} + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}x^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}x^3 + \cdots$

(ii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Students should be able to obtain, for instance, expansion of $\frac{(1+x)^{\frac{1}{3}}}{(1-x)^2}$ in ascending powers of x to a specified number of terms.

<u>Example 1</u>
Application of the binomial expansion to approximations.

<u>Example 2</u>
Application to the study of the exponential series (in Unit 3).

<u>Example 3</u>
Use the binomial expansion to find a polynomial $p(x)$ of a certain degree which approximates a certain rational function (e.g. $\frac{(1+x)^3}{(1-x)^2}$) for a specified range of values of x.